Stochastic resonance in piecewise potentials

1998 J. Phys. A: Math. Gen. 31 9773

(http://iopscience.iop.org/0305-4470/31/49/002)

J. Phys. A: Math. Gen. **31** (1998) 9773–9786. Printed in the UK PII: S0305-4470(98)95482-8

Stochastic resonance in piecewise potentials

V Berdichevsky and M Gitterman
Department of Physics, Bar-Ilan University, Ramat-Gan 52900, Israel

Received 23 June 1998

Abstract. Different characteristics of an output signal (average position, population, average energy) are calculated for a particle moving in a piecewise potential and subject to external periodic and random forces. Particular emphasis has been placed on the dependence of these characteristics on the strength of the noise and the frequency of an external field. An external periodic force strives to equalize the populations of the discrete levels, or even to reverse the populations for the space-extended systems. The populations of the potential wells in a one-barrier system subject to oscillation of the wells can either decrease or increase (compared with the field-free case) depending on the frequencies of the external field. All these changes of populations induced by an external field have some resemblance to the similar quantum mechanical problems.

1. Introduction

An investigation of dynamic systems subject to a periodic and/or random field has attracted considerable interest. The distinguishing feature of these phenomena is a non-monotonic dependence of an output signal on one or more parameters that characterize a random force or an external periodic signal. The latter are described under the common name 'stochastic resonance' (resonance activation, coherent stochastic resonance, etc) [1]. The quantitative description of these nonlinear systems usually requires numerical simulations while analytical solutions can be found only in some exceptional cases.

Recently [2–4], we have presented analytical solutions for one-dimensional diffusion of a classical particle through a bistable piecewise potential. For the time-independent problem, we considered the influence of reflecting walls and barrier heights on the transmission of the particle, the asymptotic behaviour for small and large time, comparison with the Kramers escape rate [2], as well as resonance tunnelling through a double barrier [3]. For the time-dependent problem (one potential well subject to a periodic signal), the phenomenon of stochastic resonance occurs and the signal-to-noise ratio turns out to be a non-monotonic function not only of the noise strength but also of the frequency of the external field [4].

This paper deals with the analysis of manifestations of stochastic resonance by comparing the analytical solutions for different time-dependent problems. The special form of the matching conditions described below allows us to consider a wide range of phenomena which were beyond the techniques used in [4]. This form of matching conditions is particularly suited to geometrically restricted systems with reflecting or absorbing walls and for the solutions asymptotic in time where the initial conditions have already been washed out. Although our analysis is restricted by these two limitations, we consider and compare the following different models: two-level discrete and space-extended models, systems with a step potentials with those with a barrier potential, a one-barrier potential with oscillating well and with an oscillating barrier, and systems of two wells with equal and different potential barriers.

For all the above-mentioned systems, we find the exact solutions of dynamic problems which allows us to calculate the different characteristics of an output signal, namely, both the time-independent and oscillating parts of asymptotic populations, the average energy of the output signal, and its average position in the well. Then, we check which of these characteristics show non-monotonic behaviour as a function of the noise strength and the frequency of an external field for different forms of the potential.

The general idea of our approach is illustrated in the next section by means of the simple two-state discrete model subject to an external periodic field.

2. Two-state system subject to periodic perturbations

There are many applications in science for the simplest model of a system which can be found either in the 'left', (population n_1), or in the 'right', (population n_2), states where $n_1 + n_2 = 1$.

A rate equation describing the dynamics of a two-state system has the following form:

$$\frac{\mathrm{d}n_1}{\mathrm{d}t} = -\frac{\mathrm{d}n_2}{\mathrm{d}t} = W_2 n_2 - W_1 n_1 = W_2 - (W_1 + W_2) n_1 \tag{1}$$

where $W_{1(2)}$ is the transition rate out of state 1(2). In the absence of an external field, the rates usually have the Arrhenius form, $W^0 \sim \exp(-\frac{U}{T})$, where U is the height of the potential barrier and T is the temperature (in units of energy) which sometimes is replaced by the noise intensity D if the transition between the two states is initiated by non-thermal noise. We consider the case when, say, the left state is less stable, i.e. the potential barrier U_1 for transmission to the right state is lower than U_2 for the reverse transition, $U_1 < U_2$. If we denote

$$W_1^0 \sim \exp\left(-\frac{U_1}{D}\right) \qquad W_2^0 \sim \exp\left(-\frac{U_2}{D}\right) \tag{2}$$

then, after some transient process, one gets for $t \to \infty$

$$n_{1,\infty} = \frac{W_2^0}{W_1^0 + W_2^0} \qquad n_{2,\infty} = \frac{W_1^0}{W_1^0 + W_2^0} \tag{3}$$

i.e. $n_{1,\infty} < n_{2,\infty}$—the left 'shallow' state contains fewer particles then the right 'deep' state.

The influence of the external periodic field is usually described by the modulation of the energy levels, i.e. U_1 and U_2 are replaced by $U_1 + A\cos(\Omega t)$ and by $U_2 - A\cos(\Omega t)$, respectively. Then, the positions of the potential minima will oscillate in antiphase with the period of the external field. Another possibility is that the potential barrier oscillates while the energy levels remain fixed. Then, one has to replace U_1 and U_2 by $U_1 + A\cos(\Omega t)$ and by $U_2 + A\cos(\Omega t)$, i.e. the barriers oscillate in phase.

Substituting the antiphase modulation of the wells, $U_1 + A\cos(\Omega t)$ and $U_2 - A\cos(\Omega t)$, in the rate equation (1), one can rewrite the latter as

$$\frac{\mathrm{d}n_1}{\mathrm{d}t} + \left[W_1^0 \exp\left(\frac{A\cos(\Omega t)}{D}\right) + W_2^0 \exp\left(-\frac{A\cos(\Omega t)}{D}\right)\right] n_1 = W_2^0 \exp\left(-\frac{A\cos(\Omega t)}{D}\right). \tag{4}$$

After some transient period, the initial conditions of equation (4) will be washed out, and the solution becomes periodic in time.

$$n_1 = n_{1,\infty} + \sum_m [R_m \cos(m\Omega t) + S_m \sin(m\Omega t)] = n_{1,\infty} + \sum_m \sqrt{R_m^2 + S_m^2} \sin(m\Omega t + \phi_m). \tag{5}$$

Using the expansion of $\exp(\pm\frac{A\cos(\Omega t)}{D})$ in a series of modified Bessel functions of the first kind [5] yields

$$\exp\left(\pm\frac{A\cos(\Omega t)}{D}\right) = \sum_{l\doteq-\infty}^{\infty} I_l\left(\pm\frac{A}{D}\right)\cos(l\Omega t). \tag{6}$$

Substituting both (5) and (6) into (4), one can find the recursive relations for $n_{1,\infty}$, R_m and S_m, as was done in similar problems [6, 7]. Truncating the recursive relations at $m = 0, 1, 2\ldots$, one obtains the sets of coefficients in equation (5) which corresponds to increasing powers of $\frac{A}{D}$, i.e. of the amplitude of an external field. Omitting straightforward calculations, we write the final results up to the lowest order in the field amplitude, i.e. to first order for A_1 and B_1, and in second order for $n_{1,\infty}$:

$$\begin{aligned} n_{1,\infty} &= \frac{W_2^0}{W_1^0 + W_2^0} + \frac{A^2}{D^2}\frac{W_1^0 W_2^0}{\Omega^2 + (W_1^0 + W_2^0)^2}\frac{W_1^0 - W_2^0}{W_1^0 + W_2^0} \\ \sqrt{R_1^2 + S_1^2} &= \frac{2AW_1^0 W_2^0}{D(W_1^0 + W_2^0)\sqrt{\Omega^2 + (W_1^0 + W_2^0)^2}}. \end{aligned} \tag{7}$$

In all previous analyses [6, 7, and others], the two stable states ($W_1^0 = W_2^0$) have been considered and the limiting ($t \to \infty$) values of $n_{1,\infty}$, $n_{2,\infty}$ did not change in the presence of an external periodic field. The only influence of the field was to cause a periodic change of the population of the two states described by the coefficients R_m and S_m in equation (5).

As one can see from equation (7), in the presence of a field, the field-free expression for $n_{1,\infty}$ contains an additional positive term (since $W_1^0 > W_2^0$). More positive terms will come from the next order corrections in $\frac{A}{D}$. Then, one obtains a quite unexpected result: the less stable state becomes 'more stable' in the presence of an external periodic field. In fact, this field tends not only to equalize the populations of two states: under some circumstances it can even reverse them. We will consider this phenomenon in more detail in section 5.

The second conclusion which follows from equation (7) is the behaviour of the amplitude of the oscillations, $\sqrt{R_1^2 + S_1^2}$, which is monotonic as a function of the external field frequency Ω, but non-monotonic as a function of the noise strength D.

Thus far, we have considered the oscillating potential minima. Quite different results are obtained for the oscillating barrier. In this case, for both states the potential changes are $U_{1,2} + A\cos(\Omega t)$, and all but first terms in equation (4) have the same exponential factor $\exp(\frac{A\cos(\Omega t)}{D})$. This factor can be eliminated from the equation by changing the time variable t to $\tau = \int_0^t \exp(\frac{A\cos(\Omega z)}{D})\,dz$, and one returns to equation (1) with $W_{1,2}$ replaced by $W_{1,2}^0$ and t replaced by τ. It is clear, therefore, that, in contrast to equation (7), the limit $t \to \infty$ populations are not changed in the presence of an external field.

3. Basic equations

As was shown during the last 20 years, many fundamental properties of a particle moving in a nonlinear potential under the influence of both periodic and random signals are generic, and are not too sensitive to the details of the potential. Therefore, it is worthwhile to consider the simplest potential which allows an analytic solution, in addition to carrying out numerical simulations for more complicated potentials.

We consider a particle moving in the piecewise potentials $U(x)$ under the influence of white noise. We choose simple square-well potentials with one and two barriers restricted by reflecting walls. In [2], we presented the full dynamic solution of these problems in the absence of an external periodic field. In [4], we added a periodic signal acting on the left potential well, and after some quite complicated mathematics, we found the analytical solution (for small amplitude of the external field) for the signal-to-noise ratio. Since there were no restriction on the frequency of periodic field, the new phenomenon found was the non- monotonic dependence of the signal-to-noise ratio as a function of the field frequency. In this article, we use a much simpler approach which allows a set of analytical solutions for the asymptotic limit $t \to \infty$.

The Fokker–Planck equation for the probability function $P(x, t)$ for the position x of a diffusive particle at the time t is

$$\frac{\partial P}{\partial t} = \frac{\partial}{\partial x}\left[\frac{\partial U}{\partial x}P + D\frac{\partial P}{\partial x}\right] \equiv -\frac{\partial J}{\partial x} \tag{8}$$

where the probability current J is defined in equation (8).

For the piecewise potentials, $\frac{\partial U}{\partial x} = 0$, equation (8) reduces to a simple diffusion equation. Moreover, our choice for the periodic signal does not introduce an additional force in equation (8) which still has the form of a simple diffusion equation. However, the periodic signal enters the matching conditions, namely, one has to solve equation (8) in each region of $U(x) =$ constant, and then to ensure the continuity of J across the boundaries of these regions. Continuity of probability current J, which according to equation (8) can be written as $J = -De^{-\frac{U}{D}}\frac{d}{dx}(e^{\frac{U}{D}}P)$, means that at points z of the jumps of potentials,

$$e^{\frac{U(z+0)}{D}}P(z+0, t) = e^{\frac{U(z-0)}{D}}P(z-0, t) \tag{9}$$

$$\frac{\partial P(z+0, t)}{\partial x} = \frac{\partial P(z-0, t)}{\partial x}. \tag{10}$$

The matching conditions (9) and (10) have to be complemented by reflected boundary conditions at the positions z of the walls,

$$\frac{\partial P(z, t)}{\partial x} = 0. \tag{11}$$

An external periodic signal enters the exponents in equation (9), i.e. the solutions of equation (8) will be periodic in time with the period $2\pi\Omega^{-1}$ of the external field.

Our main assumption is the smallness of the amplitude of the external field which means $\frac{A}{D} < 1$. Therefore, we use expansion (6) in equation (9), and seek the solution of equation (8) in each region m as

$$P_m = S_m^0 + \sum_{l=1}^{\infty}\left(\frac{A}{D}\right)^l f_m^{(l)}(x, t) \tag{12}$$

where $f_m^{(l)}$ is a periodic function of t which can be written in the following form:

$$f_m^{(l)} = f_{m,0}^{(l)} + \sum_{k=1}^{\infty}[(f_{m,k}^{(l)}e^{r_k x} + \tilde{f}_{m,k}^{(l)}e^{-r_k x})e^{i\Omega k t} + \text{c.c.}] \qquad r_k \equiv \sqrt{\frac{i\Omega k}{D}}. \tag{13}$$

As in the previous section, we will keep only the lowest non-vanishing corrections to the field-free asymptotic probabilities S_m in the amplitude of the external field. As we shall

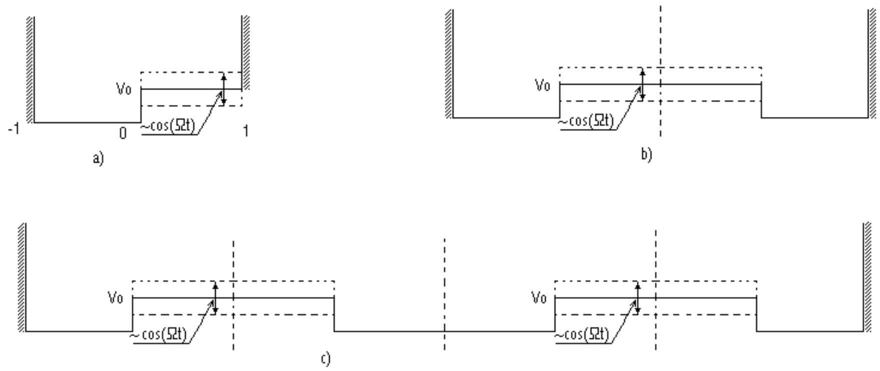

Figure 1. (*a*) One-step potential of height V_0. The right side is oscillating with frequency Ω. Both (*b*) one-barrier and (*c*) two-barrier symmetric potentials reduce to one-step potential by the use of the dotted line(s).

see, the time-independent first-order corrections in the field vanish. Therefore, we also keep the non-oscillating second-order terms in $(\frac{A}{D})$, namely,

$$P_m = S_m + \frac{A}{D}\left(f_{m,0}^{(l)} + \sum_{k=1}^{\infty}[(f_{m,k}^{(1)}\mathrm{e}^{r_k x} + \tilde{f}_{m,k}^{(1)}\mathrm{e}^{-r_k x})\mathrm{e}^{\mathrm{i}\Omega kt} + \text{c.c.}]\right) + \left(\frac{A}{D}\right)^2 f_{m,0}^{(2)}. \tag{14}$$

It will be clear later that only terms with $k = 1$ are relevant in our approximation. Further calculations depend on the form of the potential. The simplest form of the step-like potential is considered in the next section.

4. One-step potential

The form of this potential of height V_0 with two steps of equal size (called 1 and 2 from left to right) is shown in figure 1. The right side undergoes oscillations with frequency Ω. Note that in the asymptotic limit $t \to \infty$ considered here, where the initial conditions are washed out, the one-step potential is the generic form of more general potentials. Indeed, as one can see from figure 1, because of the symmetry, a one-barrier potential (with a dotted section shown in figure 1) is a set of two one-step potentials, a two-barrier potential (with two dotted sections) is a set of four one-step potentials, and so on. In all these cases, only the barriers oscillate while the wells remain immobile. In section 5, we consider a one-barrier potential with an oscillating well and immobile barrier.

The matching condition (9) at point $x = 0$ for the probability functions P_m defined in equation (14) has the following form:

$$S_1 + \frac{A}{D}\left(f_{1,0}^{(1)} + \sum_{k=1}^{\infty}[(f_{1,k}^{(1)} + \tilde{f}_{1,k}^{(1)})\mathrm{e}^{\mathrm{i}\Omega kt} + \text{c.c.}]\right) + \left(\frac{A}{D}\right)^2 f_{1,0}^{(2)}$$
$$= \left(S_2 + \frac{A}{D}\left(f_{2,0}^{(1)} + \sum_{k=1}^{\infty}[(f_{2,k}^{(1)} + \tilde{f}_{2,k}^{(1)})\mathrm{e}^{\mathrm{i}\Omega kt} + \text{c.c.}]\right) + \left(\frac{A}{D}\right)^2 f_{2,0}^{(2)}\right)\mathrm{e}^{\frac{V_0 + A\cos(\Omega t)}{D}}. \tag{15}$$

Expanding the exponential in powers of $\frac{A}{D}$ and collecting terms containing appropriate powers of the small parameter $\frac{A}{D}$, one obtains in zero order (without external field)

$$S_1 = S_2 e^{\frac{V_0}{D}}. \tag{16}$$

Adding to equation (16) the normalization condition $\int_{-1}^{1} P\,dx = 1$, which reduces to $S_1 + S_2 = 1$, one gets

$$S_1 = \frac{1}{1 + e^{-\frac{V_0}{D}}} \qquad S_2 = \frac{e^{-\frac{V_0}{D}}}{1 + e^{-\frac{V_0}{D}}}. \tag{17}$$

Collecting all terms of first order in $\frac{A}{D}$, one obtains

$$f_{1,0}^{(1)} + \sum_{k=1}^{\infty}[(f_{1,k}^{(1)} + \tilde{f}_{1,k}^{(1)})e^{i\Omega kt} + \text{c.c.}]$$
$$= e^{\frac{V_0}{D}} f_{2,0}^{(1)} + e^{\frac{V_0}{D}} \sum_{k=1}^{\infty}[(f_{2,k}^{(1)} + \tilde{f}_{2,k}^{(1)})e^{i\Omega kt} + \text{c.c.}] + S_2 e^{\frac{V_0}{D}} \cos(\Omega t). \tag{18}$$

A comparison of the time-independent terms in equation (18), together with the normalization condition which reduces to $f_{1,0}^{(1)} + f_{2,0}^{(1)} = 0$, leads to $f_{1,0}^{(1)} = f_{2,0}^{(1)} = 0$.

Collecting the coefficients of $e^{i\Omega kt}$ in equation (18), we obtain

$$f_{1,1}^{(1)} + \tilde{f}_{1,1}^{(1)} = e^{\frac{V_0}{D}}(f_{2,1}^{(1)} + \tilde{f}_{2,1}^{(1)} + S_2/2) \tag{19}$$

and an analogous equation for the complex conjugate.

Hence, only terms with $k = 1$ in the sum of equation (18) are affected by the external field in the first approximation in $\frac{A}{D}$. All other terms in k can be omitted, i.e. the probability function P_m reduces to

$$P_m = S_m + \frac{A}{D}[(f_m e^{rx} + \tilde{f}_m e^{-rx})e^{i\Omega t} + \text{c.c.}] + \left(\frac{A}{D}\right)^2 g_m. \tag{20}$$

For simplicity, we omit the upper indices of $f_m^{(l)}$ and rewrite $f_{m,0}^{(2)}$ as g_m. The remaining boundary condition (10) takes the form

$$f_1 - \tilde{f}_1 = f_2 - \tilde{f}_2 \tag{21}$$

while the reflective boundary conditions (11) at $x = -1$ and $x = 1$ are

$$\begin{aligned} f_1 e^{-r} - \tilde{f}_1 e^{r} &= 0 \\ f_2 e^{r} - \tilde{f}_2 e^{-r} &= 0. \end{aligned} \tag{22}$$

Solving the four equations (19), (21) and (22), one gets

$$f_1 = -\tilde{f}_2 = \frac{e^{\frac{V_0}{D}}}{2(1 + e^{\frac{V_0}{D}})^2(1 + e^{-2r})} f_2 = -\tilde{f}_1 = -e^{-2r} f_1. \tag{23}$$

Separating from equation (15) the terms of second order in $\frac{A}{D}$, one gets

$$g_1 = \left[\frac{S_2}{4} + \frac{1}{2}(\tilde{f}_2 + f_2 + \text{c.c.}) + g_2\right] e^{\frac{V_0}{D}}. \tag{24}$$

The normalization of the probability function was already calculated to zero-order terms in $\frac{A}{D}$. Therefore, for the second-order terms, $g_1 + g_2 = 0$. Using this latter condition and equations (23) and (24), one gets

$$g_1 = -g_2 = -\frac{e^{\frac{V_0}{D}}}{4} \frac{(e^{\frac{V_0}{D}} - 1)}{(1 + e^{\frac{V_0}{D}})^3}. \tag{25}$$

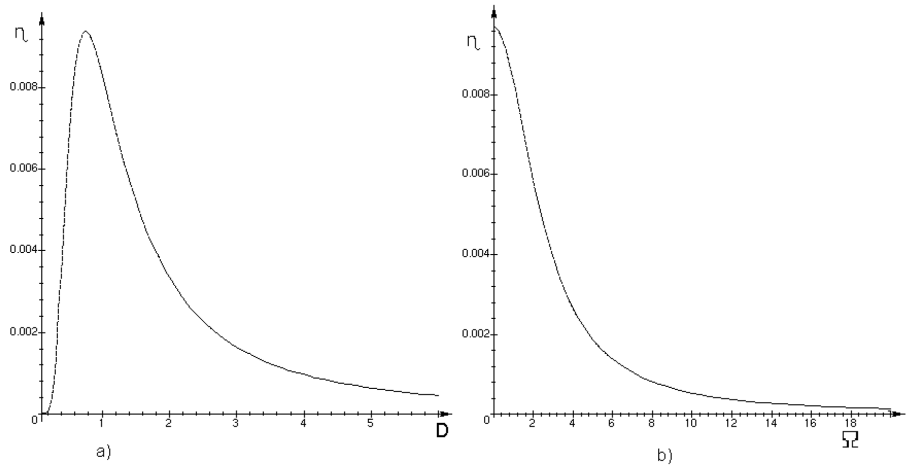

Figure 2. Amplification factor η for a one-step potential as a function of noise strength D (for $\Omega = 1$, $V_0 = 1$) and of the frequency Ω of an external field (for $D = 1$, $V_0 = 1$).

The latter results mean that the time-independent shift to the probability distribution function, which according to equation (12) is proportional to the square of the amplitude of an external field, turns out to be negative for the left well and positive for the right well, independent of frequency.

The averaged position of a particle is another interesting parameter. Using the results obtained above, one gets

$$\bar{x}(t) \equiv \int_{-1}^{1} x P(x,t)\,\mathrm{d}x = \int_{-1}^{0} x P_1(x,t)\,\mathrm{d}x + \int_{0}^{1} x P_2(x,t)\,\mathrm{d}x = \bar{x}_0 + B\cos(\Omega t + \varphi) \quad (26)$$

$$B = \frac{A}{D}\frac{2D\mathrm{e}^{\frac{V_0}{D}}}{\Omega(1+\mathrm{e}^{\frac{V_0}{D}})^2}\frac{\cosh\left(\sqrt{\frac{\Omega}{8D}}\right)^2 - \cos\left(\sqrt{\frac{\Omega}{8D}}\right)^2}{\left(\cosh\left(\sqrt{\frac{\Omega}{2D}}\right)^2 - \sin\left(\sqrt{\frac{\Omega}{2D}}\right)^2\right)^{1/2}}. \quad (27)$$

The first term in equation (26) describes the average position of a particle, whereas the second term describes the periodic changes with the period of an external field.

The amplitude of the oscillations shows the non-monotonic dependence on the noise strength and the monotonic dependence on the frequency of the external signal. In figure 2 we show the amplification parameter $\eta \equiv \frac{|B|^2}{A^2}$ as a function of the noise strength D, which is a manifestation of stochastic resonance, and of the frequency Ω of an external signal.

In contrast to the amplification factor, there exists another characteristic of the averaged motion which shows non-monotonic behaviour both in D and Ω, namely, the average energy $\langle E\rangle$ of a signal which is proportional to $\langle(\frac{\mathrm{d}\bar{x}}{\mathrm{d}t})^2\rangle$, i.e. to $\frac{B^2\Omega^2}{2}$. As one can see in figure 3, the dependence of the averaged energy on D is much more profound than that on Ω.

Another function of interest is the probability $W(t)$ to find a particle in one of the regions of the step potential which is equivalent to populations in a two-level problem described in

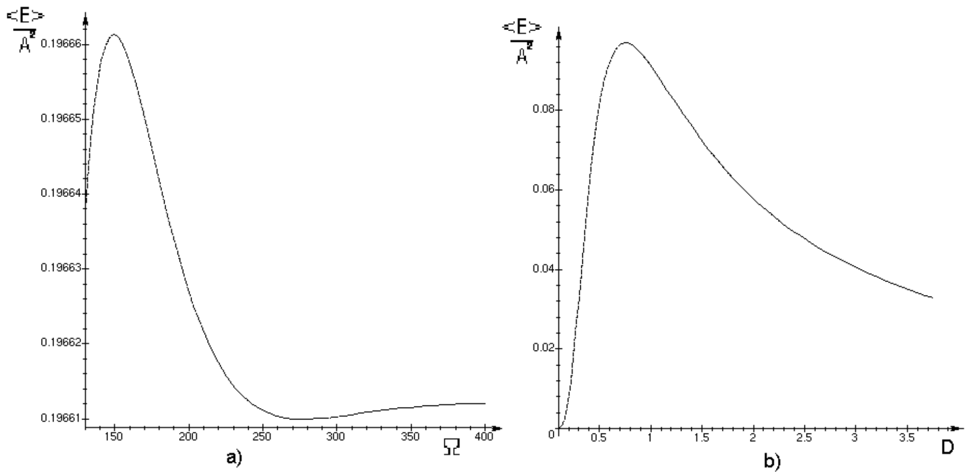

Figure 3. Average energy of an output signal for a one-step potential as a function of (*a*) the frequency Ω of an external field (for $D = 1$, $V_0 = 1$) and (*b*) the noise strength D (for $\Omega = 1$, $V_0 = 1$).

section 2. Hence, in the region $[-1, 0]$

$$\begin{aligned} W(t) \equiv \int_{-1}^{0} P_1(x,t)\,\mathrm{d}x &= S_1 + \left(\frac{A}{D}\right)^2 g_1 + \frac{A}{D}\left[\frac{(1-\mathrm{e}^{-2r})}{r} f_1 \mathrm{e}^{\mathrm{i}\Omega t} + \text{c.c.}\right] \\ &= \frac{1}{1+\mathrm{e}^{-\frac{V_0}{D}}} - \left(\frac{A}{D}\right)^2 \frac{\mathrm{e}^{\frac{V_0}{D}}}{4} \frac{(\mathrm{e}^{\frac{V_0}{D}}-1)}{(1+\mathrm{e}^{\frac{V_0}{D}})^3} \\ &\quad + \frac{A}{D}\left[\frac{\mathrm{e}^{\frac{V_0}{D}}}{2}\frac{\tanh(2r)}{r(1+\mathrm{e}^{\frac{V_0}{D}})^2}\mathrm{e}^{\mathrm{i}\Omega t} + \text{c.c.}\right]. \end{aligned} \tag{28}$$

The external field corrections in equation (28) that contain the square of the amplitude are negative, which means that the population of the more populated state is decreasing, analogously to the two-level system described in section 2. Likewise, the amplitude of the time-dependent corrections that is linear in $\frac{A}{D}$ is a monotonic function of the external field frequency Ω, but non-monotonic as a function of the noise strength D.

As it was emphasized above, because of the symmetry, the solutions (26), (27) and (28) obtained for a step potential are, at the same time, solutions for one-, two- and many-barrier potentials with oscillating barrier(s).

5. One-barrier potential with oscillating well

Figure 4 shows the square-well potential with the periodic signal acting on the left well. For simplicity we consider symmetric barriers with two wells of equal size but with different potential heights U_1 and U_2.

The general method of solving the Fokker–Planck equation (8) for this case is very similar to that used in the previous section for the step-like potential. The three different regions in figure 4 with two matching conditions at $x = 0$ and $x = 1$, instead of two regions

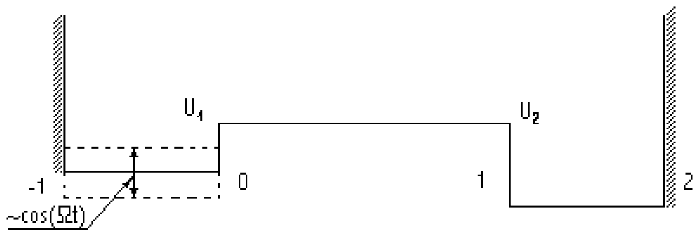

Figure 4. Square double-well potential with oscillating left well.

and one matching condition point in figure 1, make the solution slightly more cumbersome, although the general matching equations (9) and (10) as well as the boundary conditions (equation (11)) are the same in both cases.

We are seeking the solution of the Fokker–Planck equation in the form (20).

For zero-order in $\frac{A}{D}$ terms in (20), the matching conditions at $x = 0$ and $x = 1$ are

$$S_1 = S_2 e^{\frac{U_1}{D}} \qquad S_3 = S_2 e^{\frac{U_2}{D}}. \tag{29}$$

Adding to (29) the normalization condition:

$$1 = \int_{-1}^{0} S_1 \, dx + \int_0^1 S_2 \, dx + \int_1^2 S_3 \, dx = S_1 + S_2 + S_3 \tag{30}$$

one immediately obtains from (29) and (30):

$$S_1 = \frac{e^{\frac{U_1}{D}}}{1 + e^{\frac{U_1}{D}} + e^{\frac{U_2}{D}}} \qquad S_2 = \frac{1}{1 + e^{\frac{U_1}{D}} + e^{\frac{U_2}{D}}} \qquad S_3 = \frac{e^{\frac{U_2}{D}}}{1 + e^{\frac{U_1}{D}} + e^{\frac{U_2}{D}}}. \tag{31}$$

Repeating the analysis of the previous section, one finds that for the terms that are first-order in $\frac{A}{D}$, only terms with $k = 1$ in the sum are relevant to our problem, and the probability distribution function in each of the regions 1–3 has the form (20). Using the matching conditions (9) and (10) for this function, one has at $x = 0$ to first order in $\frac{A}{D}$

$$\begin{aligned} & f_1 - \tilde{f}_1 = f_2 - \tilde{f}_2 \\ & \frac{S_1}{2} + f_1 + \tilde{f}_1 = (f_2 + \tilde{f}_2) e^{\frac{U_1}{D}} \end{aligned} \tag{32}$$

and at $x = 1$:

$$\begin{aligned} & f_2 e^{r} - \tilde{f}_2 e^{-r} = f_3 e^{r} - \tilde{f}_3 e^{-r} \\ & (f_2 e^{r} + \tilde{f}_2 e^{-r}) e^{\frac{U_2}{D}} = f_3 e^{r} + \tilde{f}_3 e^{-r}. \end{aligned} \tag{33}$$

Finally, the reflecting boundary conditions (11) at the walls $x = -1$ and $x = 2$ are

$$\begin{aligned} & f_1 e^{-r} - \tilde{f}_1 e^{r} = 0 \\ & f_3 e^{2r} - \tilde{f}_3 e^{-2r} = 0. \end{aligned} \tag{34}$$

Solving the six equations (32)–(34) for the six variables f_1, $\tilde{f}_1$, f_2, $\tilde{f}_2$, f_3 and $\tilde{f}_3$, one gets

$$\begin{aligned} f_1 &= -(e^{2r}+1)(e^{\frac{U_2}{D}}+1)K \\ \tilde{f}_1 &= -(e^{2r}+1)(e^{\frac{U_2}{D}}+1)e^{-2r}K \\ f_2 &= -[e^{-2r}(e^{\frac{U_2}{D}}+1)+(1-e^{\frac{U_2}{D}})]K \\ \tilde{f}_2 &= [e^{2r}(e^{\frac{U_2}{D}}+1)+(1-e^{\frac{U_2}{D}})]K \\ f_3 &= 2e^{\frac{U_2}{D}}e^{-2r}K \\ \tilde{f}_3 &= 2e^{\frac{U_2}{D}}e^{2r}K \\ K &\equiv \frac{1}{8}\frac{e^{\frac{U_1}{D}}}{(1+e^{\frac{U_1}{D}}+e^{\frac{U_2}{D}})(1+e^{\frac{U_1}{D}}+e^{\frac{U_2}{D}}+e^{\frac{U_1+U_2}{D}}\tanh^2(r))\cosh^2(r)}. \end{aligned} \tag{35}$$

The matching conditions to second order in $\frac{A}{D}$ have the simple form $g_3 = g_2 e^{\frac{U_2}{D}}$ at $x = 1$, and a more complicated form at $x = 0$,

$$e^{-\frac{U_1}{D}}\left[\frac{S_1}{4}+\mathrm{Re}\,(f_1+\tilde{f}_1)+g_1\right] = g_2. \tag{36}$$

Adding to these equations the normalization condition $g_1 + g_2 + g_3 = 1$, and using equation (35), one obtains

$$\begin{aligned} g_1 &= -(S_2+S_3)\left[\frac{S_1}{4}+\mathrm{Re}\,(f_1+\bar{f}_1)\right] \\ g_2 &= -\frac{g_1}{1+e^{\frac{U_2}{D}}} \qquad g_3 = -\frac{g_1}{1+e^{-\frac{U_2}{D}}} \end{aligned} \tag{37}$$

where

$$\begin{aligned} \mathrm{Re}\,(f_1+\tilde{f}_1) &= -\frac{1+e^{-\frac{U_2}{D}}}{2e^{\frac{U_1+U_2}{D}}}\,\frac{H+\frac{\sin^2(a_2)\sinh^2(a_2)}{4H}}{H-\frac{\sinh^2(a_2)-\sin^2(a_2)}{4}} \\ H &\equiv \frac{(1+e^{\frac{U_1}{D}}+e^{\frac{U_2}{D}})[\sinh^2(a_1)+\cos^2(a_1)]^2}{e^{\frac{U_1}{D}}+e^{\frac{U_2}{D}}}+\frac{\sinh^2(a_2)-\sin^2(a_2)}{4} \\ a_2^n &\equiv \sqrt{\frac{n^2\Omega}{2D}}. \end{aligned}$$

Equations (35) and (38) present a full solution of our problem to second order in $\frac{A}{D}$. We are able now to analyse all the physical quantities of interest.

Let us compare the time-independent probabilities (to $(\frac{A}{D})^2$) for the left, $n_{1,\infty}$, and the right, $n_{2,\infty}$, wells which are equivalent to populations in the two-level problem considered in section 2:

$$\begin{aligned} n_{1,\infty} &= \int_{-1}^{0} P_1(x,t)\,dx = S_1 + \left(\frac{A}{D}\right)^2 g_1 \\ n_{2,\infty} &= \int_{-1}^{0} P_3(x,t)\,dx = S_3 + \left(\frac{A}{D}\right)^2 g_3. \end{aligned} \tag{38}$$

If we assume, as shown in figure 4, that without an external field the left well had a lower barrier than the right well, $U_1 < U_2$, one can reverse the population with the help of an external periodic field. This will occur when $n_{1,\infty} > n_{2,\infty}$ which can be rewritten as

$$S_3 - S_1 < \left(\frac{A}{D}\right)^2 (g_1 - g_3). \tag{39}$$

Note that for the last equation, we can take values of $\tilde{f}_1$, $\tilde{f}_1, S_1$ and S_3 from equations (31), (35) and (37), which gives

$$\left(\frac{A}{D}\right)^2 \frac{1 + 2\mathrm{e}^{\frac{U_2}{D}}}{\mathrm{e}^{\frac{U_2}{D}} - \mathrm{e}^{\frac{U_1}{D}}} \left[-\mathrm{Re}\,(f_1 + \bar{f}_1) - \frac{S_1}{4}\right] > 1. \tag{40}$$

The last inequality is satisfied when the factor in front of the brackets is large, and the expression in brackets is positive. The former occurs when $A > \sqrt{U_2 - U_1}$, and the latter holds for not too small frequencies.

Thus, we have obtained an interesting result: an external periodic force acting on the 'shallow' well is able to transform it into the 'deep' well. This result has been obtained analytically for a small periodic force and consequently only for very near minima. However, one can expect that a stronger periodic signal would be able to reverse even more distant minima.

In order to compare the results obtained in this section for the oscillating well with those obtained in the previous section for the oscillating barrier, one has to consider two barriers with equal heights, i.e. to put $U_1 = U_2$ in equations (35) and (38).

The essential difference between these two cases is that in the oscillating well case, the second-order correction g_1 to the probability distribution function for the left well, in contrast to equation (25), depends on the frequency Ω of an external field. Moreover, as one can see from figure 5, this dependence is non-monotonic, reaching maxima and minima at some frequencies, all of which has an interesting analogy with quantum systems.

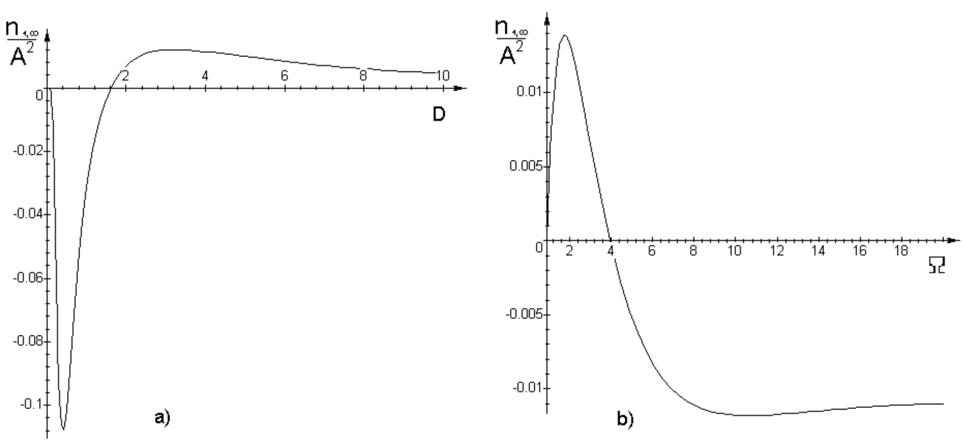

Figure 5. Field-dependent part of population of the left well as a function of (*a*) the noise strength D (for $\Omega = 100$; $U_1 = 2$; $U_2 = 3$) and (*b*) the frequency Ω of an external field (for $D = 0.7$; $U_1 = U_2 = 1$).

The population in the left well, $n_{1,\infty}(t)$, has the following form:

$$n_{1,\infty}(t) \equiv \int_{-1}^{0} P_1(x,t)\,\mathrm{d}x = S_1 + \left(\frac{A}{D}\right)\left|\frac{\bar{f}_1(\mathrm{e}^r - 1)}{r}\right|\cos(\Omega t + \phi) + \left(\frac{A}{D}\right)^2 g_1. \tag{41}$$

The non-monotonic dependence on frequency of the correction term which is second order in the field $((\frac{A}{D})^2 g_1$ in equation (41)) has a resemblance to the enhancement of tunnelling versus localization in quantum systems subject to an external field of different frequencies [8].

The last quantity of interest is the average position of a particle $\bar{x}(t)$. For the case considered here of an oscillating left well, one obtains

$$\bar{x}(t) \equiv \int_{-1}^{2} x P(x,t)\,\mathrm{d}x = \bar{x}_0 + \frac{A}{D}|Z|\cos(\Omega t + \varphi) \tag{42}$$

where

$$x_0 = -\tfrac{1}{2}(S_1 + g_1) + \tfrac{1}{2}(S_2 + g_2) + \tfrac{3}{2}(S_3 + g_3)$$
$$Z = -\frac{16\bar{f}_1 \mathrm{e}^r \sinh^2(\frac{r}{2})[\mathrm{e}^{\frac{U_2}{D}}\cosh^2(\frac{r}{2}) + \frac{1}{2}\cosh(r)]}{r^2\cosh(r)(\mathrm{e}^{\frac{U_2}{D}} + 1)}. \tag{43}$$

The amplitude of the oscillations in equation (42) behaves monotonically with Ω and shows the usual stochastic resonance as a function of the noise strength D (figure 6). Analogous to the results obtained in the previous section, the average energy $\langle E\rangle$ is a non-monotonic function of both D and Ω.

All the non-monotonic behaviour mentioned above are manifestations of stochastic resonance which was previously analysed in terms of the so-called signal-to-noise ratio [4].

6. Summary

The distinctive characteristic of stochastic resonance (SR) is the non-monotonic behaviour of the output signal-to-noise ratio (SNR) as a function of the noise strength D for nonlinear systems subject to an external periodic field and random signal [1]. SNR can also be a non-monotonic function of the frequency Ω of an external field [4].

In a broad sense, SR means a non-monotonic dependence of different characteristics of an output signal as a function of Ω and/or D. It is precisely these characteristics that are of interest in our study.

Nonlinear features of SR pose major problems for the theoretical analysis of SR. A cure for this difficulty is to choose the simplest geometry which still shows SR. We have chosen the piecewise potential, for which the asymptotic $t \to \infty$ behaviour for the field-free case can be calculated exactly, and the corrections in the field amplitude are found in lowest orders of perturbation theory.

We have started, however, with a two-level discrete systems where, in contrast to the usual approach, the probabilities of transitions $1 \to 2$ and $2 \to 1$ are different, i.e. in the absence of an external field, the two levels have non-equal populations as $t \to \infty$. It turns out that, in addition to periodic changes of populations, an external periodic field strives for equalization of the populations of two levels as $t \to \infty$, stabilizing the lower 'metastable' level.

This interesting result is supported in the last section by the analytical solution of the space-extended one-barrier system with two wells of different heights. For this case, the periodic signal acting on the left well is able to reverse the populations of wells. This

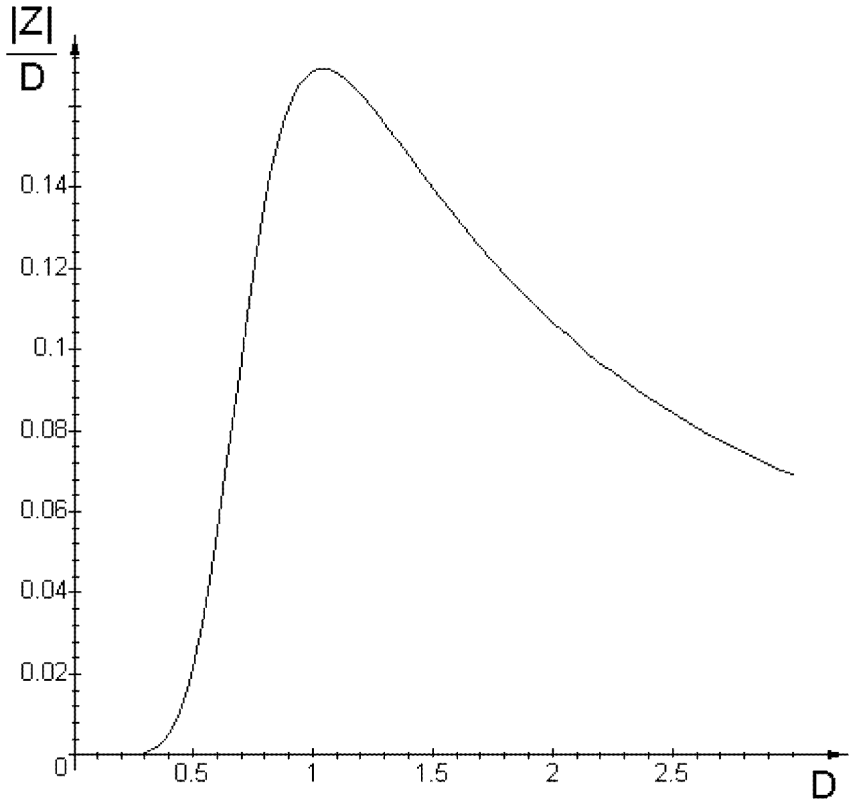

Figure 6. Amplitude of the oscillation of the average position of a particle in the square-well potential (divided by A) as a function of noise strength D (for $\Omega = 0.1$; $U_1 = 3$; $U_2 = 2$).

phenomenon is somewhat similar to the well known population inversion in quantum mechanical problems. Looking for the analogy to the phenomenon described above in classical mechanics one can mention the 'Kapitza pendulum' whose point of support oscillates vertically which stabilizes the vertically upward position [9]

The main part of this work is devoted to one-barrier systems with an oscillating barrier and with an oscillating well. We calculated the average position of the output signal and the probability to be in one of the wells, which behaves similarly for these two cases being monotonic as a function of the external field frequency Ω, but non-monotonic as a function of the noise strength D.

At the same time, we found another characteristic of the output signal: the average energy of a signal that shows non-monotonic dependence on both D and Ω.

The interesting difference occurs between the case of the oscillating barrier and that of the oscillating well. In the former case the second- order correction in the field amplitude to the populations is independent of Ω while in the latter case it shows the non-monotonic behaviour as a function of Ω. Thus, for a field of some frequencies the tunnelling through the barrier is enhanced, while for others the system becomes more localized in the presence of a field which has some resemblance to the quantum tunnelling [8].

Our analysis extends the range of studies and possible applications of the intriguing phenomenon of stochastic resonance.

References

[1] Gammaitoni L, Hanggi P, Jung P and Marchesoni F 1998 *Rev. Mod. Phys.* **70** 223
[2] Berdichevsky V and Gitterman M 1996 *J. Phys. A: Math. Gen.* **29** 1567
[3] Berdichevsky V and Gitterman M 1996 *Phys. Rev.* E **53** 1250
[4] Berdichevsky V and Gitterman M 1996 *J. Phys. A: Math. Gen.* **29** 1447
[5] Abramovitz M and Stegun I A 1970 *Handbook of Mathematical Functions* 9th edn (New York: Dover)
[6] McNamara B and Wiesenfeld K 1989 *Phys. Rev.* A **39** 4854
[7] Gitterman M and Kiefer J 1993 *Physica* A **200** 256
[8] Grifoni M and Hanggi P 1998 *Phys. Rep.* **304** 229
[9] Landau L and Lifshitz E 1969 *Mechanics* 2nd edn (Oxford: Pergamon) p 95